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Extending the Jump Analysis for Aerodynamic Asymmetry

by Gene R. Cooper

ARL-TR-3265 August 2004

NOTICES

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Army Research Laboratory
Aberdeen Proving Ground, MD 21005-5066

ARL-TR-3265 **August 2004**

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Report Documentation Page

Form Approved
OMB No. 0704-0188

1. REPORT DATE *(DD-MM-YYYY)*
August 2004

2. REPORT TYPE
Final

3. DATES COVERED (From - To)
November 2003–February 2004

4. TITLE AND SUBTITLE
Extending the Jump Analysis for Aerodynamic Asymmetry

5a. CONTRACT NUMBER

5b. GRANT NUMBER

5c. PROGRAM ELEMENT NUMBER

6. AUTHOR(S)
Gene R. Cooper

5d. PROJECT NUMBER
622618.H8011

5e. TASK NUMBER

5f. WORK UNIT NUMBER

7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES)
U.S. Army Research Laboratory
ATTN: AMSRD-ARL-WM-BC
Aberdeen Proving Ground, MD 21005-5066

8. PERFORMING ORGANIZATION REPORT NUMBER
ARL-TR-3265

9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)

10. SPONSOR/MONITOR'S ACRONYM(S)

11. SPONSOR/MONITOR'S REPORT NUMBER(S)

12. DISTRIBUTION/AVAILABILITY STATEMENT
Approved for public release; distribution is unlimited.

13. SUPPLEMENTARY NOTES

14. ABSTRACT
The linear theory for spinning projectiles was extended to account for aerodynamic asymmetry of a free-flight projectile with varying roll rate. Assuming roll is caused by differentially canted controls allowed the rolling motion to be governed by two parameters. Aerodynamic jump resulting from the asymmetry was studied as a function of these parameters and was found to be represented in closed form by using confluent hypergeometric functions. A simple rational approximation is given and compared, to the closed-form solution, over a pertinent range of the governing parameters. Inquiries regarding jump due to asymmetry are addressed and compared to previous results which are limited in their formulation since the closed-form solution was apparently unknown.

15. SUBJECT TERMS
asymmetry, linear theory, aerodynamic jump

16. SECURITY CLASSIFICATION OF:			17. LIMITATION OF ABSTRACT	18. NUMBER OF PAGES	19a. NAME OF RESPONSIBLE PERSON
a. REPORT	b. ABSTRACT	c. THIS PAGE			Gene R. Cooper
UNCLASSIFIED	UNCLASSIFIED	UNCLASSIFIED	UL	26	19b. TELEPHONE NUMBER (*Include area code*) 410-278-3684

Standard Form 298 (Rev. 8/98)
Prescribed by ANSI Std. Z39.18

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1. Projectile Dynamic Model

This report extends the work of Murphy and Bradley[1] and Fansler and Schmidt[2] describing jump caused by slight configuration asymmetries. Some of their results are briefly repeated here, for completeness, followed by a presentation of a closed-form analytic solution. The discussion then continues with a simple rational approximation, which is then compared to the analytic solution.

Flight mechanics of most projectile configurations can be captured using a rigid body six degrees of freedom dynamic model.[3] The degrees of freedom are three position components of the projectile mass center and three Euler orientation angles of the body. Figures 1 and 2 show two helpful schematics so that the degrees of freedom are seen to be related according to the following equations of motion,[4]

$$\begin{Bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{Bmatrix} = \begin{bmatrix} c_\theta c_\psi & s_\phi s_\theta c_\psi - c_\phi s_\psi & c_\phi s_\theta c_\psi + s_\phi s_\psi \\ c_\theta s_\psi & s_\phi s_\theta s_\psi + c_\phi c_\psi & c_\phi s_\theta s_\psi - s_\phi c_\psi \\ -s_\theta & s_\phi c_\theta & c_\phi c_\theta \end{bmatrix} \begin{Bmatrix} u \\ v \\ w \end{Bmatrix}, \tag{1}$$

$$\begin{Bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{Bmatrix} = \begin{bmatrix} 1 & s_\phi t_\theta & c_\phi t_\theta \\ 0 & c_\phi & -s_\phi \\ 0 & s_\phi / c_\theta & c_\phi / c_\theta \end{bmatrix} \begin{Bmatrix} p \\ q \\ r \end{Bmatrix} = \begin{bmatrix} 1 & 0 & t_\theta \\ 0 & 1 & 0 \\ 0 & 0 & 1/c_\theta \end{bmatrix} \begin{Bmatrix} p \\ \tilde{q} \\ \tilde{r} \end{Bmatrix}, \tag{2}$$

$$\begin{Bmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{Bmatrix} = \begin{Bmatrix} X/m \\ Y/m \\ Z/m \end{Bmatrix} - \begin{bmatrix} 0 & -r & q \\ r & 0 & -p \\ -q & p & 0 \end{bmatrix} \begin{Bmatrix} u \\ v \\ w \end{Bmatrix}, \tag{3}$$

and

$$\begin{Bmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{Bmatrix} = I^{-1} \left[\begin{Bmatrix} L \\ M \\ N \end{Bmatrix} - \begin{Bmatrix} 0 & -r & q \\ r & 0 & -p \\ -q & p & 0 \end{Bmatrix} I \begin{Bmatrix} p \\ q \\ r \end{Bmatrix} \right]. \tag{4}$$

---

[1] Murphy, C. H.; Bradley, J. W. *Jump Due to Aerodynamic Asymmetry of a Missile With Varying Roll Rate*; BRL-R-1077; U.S. Army Ballistic Research Laboratory: Aberdeen Proving Ground, MD, 1959.

[2] Fansler, K. S.; Schmidt, E. M. Trajectory Perturbations of Asymmetric Fin-Stabilized Projectiles Caused by Muzzle Blast. *Journal of Spacecraft and Rockets* **1978**, *15* (1), 62–64.

[3] McCoy, R. L. *Modern Exterior Ballistics: the Launch and Flight Dynamics of Symmetric Projectiles*; Schiffer Publishing Ltd.: Atglen, PA, 1999.

[4] Murphy, C. H. *Free Flight Motion of Symmetric Missiles*; BRL-TR-1216; U.S. Army Ballistic Research Laboratory: Aberdeen Proving Ground, MD, 1963.

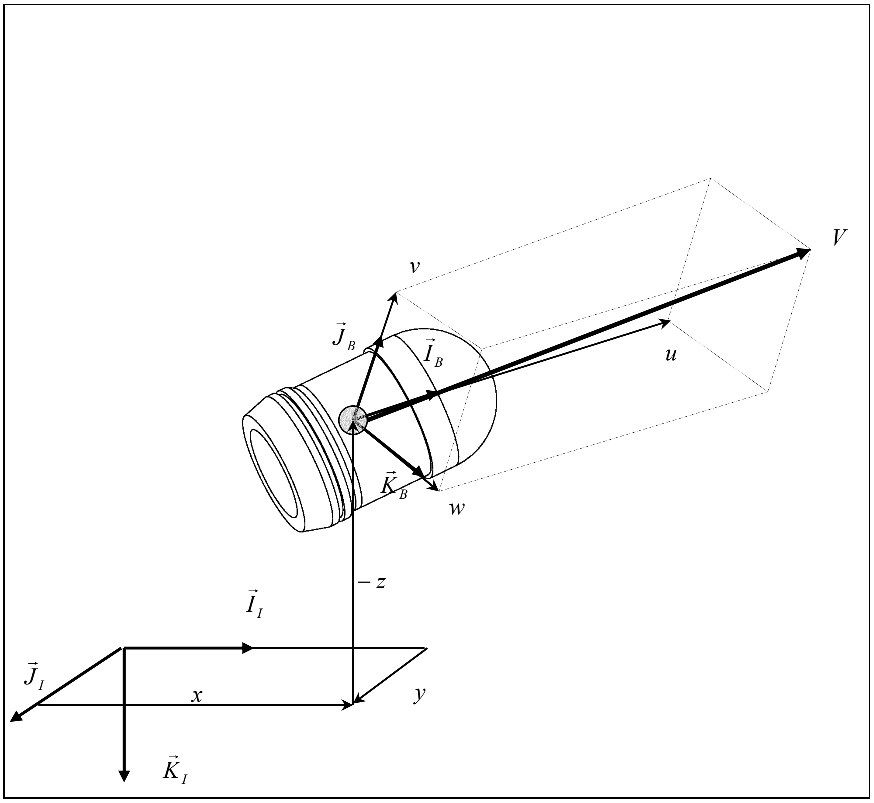

Figure 1. Projectile position coordinates definition.

Forces in the body frame that appear in equation 3 contain contributions from weight (W) and air loads (A),

$$\begin{Bmatrix} \mathrm{X} \\ \mathrm{Y} \\ \mathrm{Z} \end{Bmatrix} = \begin{Bmatrix} \mathrm{X_W} \\ \mathrm{Y_W} \\ \mathrm{Z_W} \end{Bmatrix} + \begin{Bmatrix} \mathrm{X_A} \\ \mathrm{Y_A} \\ \mathrm{Z_A} \end{Bmatrix}. \tag{5}$$

The weight force resolved into projectile body coordinates is given by equation 6,

$$\begin{Bmatrix} X_W \\ Y_W \\ Z_W \end{Bmatrix} = mg \begin{Bmatrix} -s_\theta \\ s_\phi c_\theta \\ c_\phi c_\theta \end{Bmatrix}. \tag{6}$$

The air loads are split into two components: the standard aerodynamic forces and the Magnus forces,

$$\begin{Bmatrix} \mathrm{X_A} \\ \mathrm{Y_A} \\ \mathrm{Z_A} \end{Bmatrix} = \begin{Bmatrix} \mathrm{X_{AS}} \\ \mathrm{Y_{AS}} \\ \mathrm{Z_{AS}} \end{Bmatrix} + \begin{Bmatrix} \mathrm{X_{AM}} \\ \mathrm{Y_{AM}} \\ \mathrm{Z_{AM}} \end{Bmatrix}. \tag{7}$$

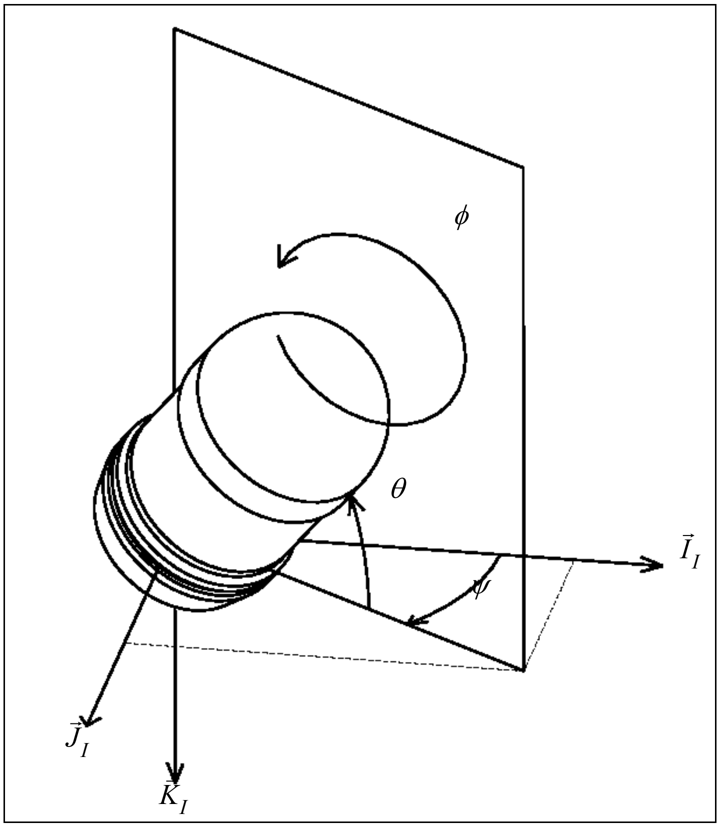

Figure 2. Projectile orientation definition.

Equation 8 gives the standard air loads acting at the aerodynamic center of pressure,

$$\begin{Bmatrix} X_{AS} \\ Y_{AS} \\ Z_{AS} \end{Bmatrix} = -q_a \begin{Bmatrix} C_{X0} + C_{X2} \dfrac{\left(v^2 + w^2\right)}{V^2} \\ C_{NA} \dfrac{v}{V} \\ C_{NA} \dfrac{w}{V} \end{Bmatrix}, \tag{8}$$

where

$$q_a = \frac{1}{8} \rho (u^2 + v^2 + w^2) \pi D^2 \tag{9}$$

and

$$V = \sqrt{u^2 + v^2 + w^2} \,. \tag{10}$$

The Magnus aerodynamic force acts at the Magnus center of pressure,

$$\begin{Bmatrix} X_{AM} \\ Y_{AM} \\ Z_{AM} \end{Bmatrix} = -q_a \begin{Bmatrix} 0 \\ \dfrac{p D C_{NPA} w}{2V} \\ \dfrac{-p D C_{NPA} v}{2V} \end{Bmatrix}. \tag{11}$$

Moments about the projectile mass center are due to aerodynamic forces, and moments (A) are

$$\begin{Bmatrix} L \\ M \\ N \end{Bmatrix} = \begin{Bmatrix} L_{SA} \\ M_{SA} \\ N_{SA} \end{Bmatrix} + \begin{Bmatrix} L_{UA} \\ M_{UA} \\ N_{UA} \end{Bmatrix}. \tag{12}$$

The aerodynamic moments caused by standard and Magnus air loads are computed with a cross product between the distance vector from the mass center to the force application point and the force itself. An unsteady aerodynamic damping moment is also present, which provides a damping source for angular motion,

$$\begin{Bmatrix} L_{UA} \\ M_{UA} \\ N_{UA} \end{Bmatrix} = q_a D \begin{Bmatrix} C_{DD} + \dfrac{pDC_{LP}}{2V} \\ \dfrac{qDC_{MQ}}{2V} \\ \dfrac{rDC_{MQ}}{2V} \end{Bmatrix}. \tag{13}$$

All aerodynamic coefficients and the center of pressures are a function of the Mach number of the projectile mass center. The dynamic model previously described is nonlinear due to both three-dimensional rotational kinematics expressions and the presence of complex aerodynamic forces.

2. Reduction to Linear Theory

Useful performance data regarding trajectory prediction and the stability of projectiles forced early ballisticians to investigate mathematical simplifications to the equations of motion. Over time, a set of simplified and solvable, yet accurate, linear differential equations emerged, which today is commonly termed “projectile linear theory.”

The governing equations previously developed are expressed in the body reference frame. In linear theory, the lateral, translational, and rotational velocity components are transformed to a nonrolling reference frame. The nonrolling frame, or so-called fixed plane frame, proceeds with only precession and nutation rotations from an inertial reference frame. Components of linear

and angular body velocities in the fixed plane frame can be computed from the body frame components of the same vector through a single-axis rotational transformation. For example, the body frame components of the projectile mass center velocity are transformed to the fixed plane frame by

$$\begin{Bmatrix} \tilde{u} \\ \tilde{v} \\ \tilde{w} \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix} \begin{Bmatrix} u \\ v \\ w \end{Bmatrix}. \tag{14}$$

Note that the $\tilde{\vdots}$ superscript indicates the vector components relative to the fixed plane reference frame. Projectile linear theory makes a change of variables from station line velocity component, u, to total velocity, V, as described in the next two equations:

$$V = \sqrt{u^2 + v^2 + w^2} = \sqrt{u^2 + \tilde{v}^2 + \tilde{w}^2}\,, \tag{15}$$

and

$$\frac{dV}{dt} = \frac{u\dfrac{du}{dt} + v\dfrac{dv}{dt} + w\dfrac{dw}{dt}}{V} = \frac{u\dfrac{du}{dt} + \tilde{v}\dfrac{d\tilde{v}}{dt} + \tilde{w}\dfrac{d\tilde{w}}{dt}}{V}. \tag{16}$$

A further change of variable from time, t, to dimensionless arc length, s, is also preferred and following Murphy[4] gives the dimensionless arc length,

$$s = \frac{1}{D}\int_0^t V dt\,. \tag{17}$$

Equations 18 and 19 relate time and arc length derivatives of a given quantity ζ. Dotted terms refer to time derivatives, and primed terms denote dimensionless arc length derivatives,

$$\dot{\zeta} = \left(\frac{D}{V}\right)\zeta', \tag{18}$$

and

$$\ddot{\zeta} = \left(\frac{D}{V}\right)^2\left[\zeta'' + \frac{V'}{V}\zeta'\right]. \tag{19}$$

Linear theory makes several assumptions regarding the relative size of different quantities to further simplify the analysis. Euler angles are small so $\sin(\theta) \approx \theta$, $\cos(\theta) \approx 1$, $\sin(\psi) \approx \psi$, and $\cos(\psi) \approx 1$, and the aerodynamic angle of attack is small so that $\alpha = \tilde{w}/V$ and $\beta = \tilde{v}/V$. The projectile is mass-balanced such that $I_{XY} = I_{XZ} = I_{YZ} = 0$ and $I_{ZZ} = I_{YY} \Rightarrow I_{YY} = I_{ZZ} \equiv I_Y$. Quantities V and ϕ are large compared to θ, ψ, q, r, v, and w, such that products of small quantities and their derivatives are negligible. Application of these assumptions results in

$$x' = D, \tag{20}$$

$$y' = \frac{D}{V}\tilde{v} + D\psi, \tag{21}$$

$$z' = \frac{D}{V}\tilde{w} - D\theta, \tag{22}$$

$$\phi' = \frac{D}{V}p, \tag{23}$$

$$\theta' = \frac{D}{V}\tilde{q}, \tag{24}$$

$$\psi' = \frac{D}{V}\tilde{r}, \tag{25}$$

$$V' = -\frac{\rho S D C_{XO}}{2m}V - \frac{Dg\theta}{V}, \tag{26}$$

$$p' = \frac{\rho S D^2 C_{LDD}}{2I_X}V + \frac{\rho S D^3 C_{LP}}{4I_X}p, \tag{27}$$

$$\begin{Bmatrix} \tilde{v}' \\ \tilde{w}' \\ \tilde{q}' \\ \tilde{r}' \end{Bmatrix} = \begin{bmatrix} -A & 0 & 0 & -D \\ 0 & -A & D & 0 \\ B/D & C/D & E & -F \\ -C/D & B/D & F & E \end{bmatrix} \begin{Bmatrix} \tilde{v} \\ \tilde{w} \\ \tilde{q} \\ \tilde{r} \end{Bmatrix} + \begin{Bmatrix} FF\, V\cos(\phi + \phi_B) \\ FF\, V\sin(\phi + \phi_B) + G \\ -MM\, V\sin(\phi + \phi_B)/D \\ MM\, V\cos(\phi + \phi_B)/D \end{Bmatrix}, \tag{28}$$

and

$$\begin{Bmatrix} A \\ B \\ C \\ E \\ F \end{Bmatrix} = \begin{Bmatrix} \dfrac{\pi\rho D^3 C_{NA}}{8\,m} \\ \dfrac{\pi\rho p D^5 C_{YPA}(SL_{MAG} - SL_{CG})}{16\, I_y V} \\ \dfrac{\pi\rho D^4 C_{NA}(SL_{COP} - SL_{CG})}{8 I_y} \\ \dfrac{\pi\rho D^5 C_{MQ}}{16\, I_y} \\ \dfrac{p D I_x}{I_y\, V} \end{Bmatrix}. \tag{29}$$

Aside from the fact that V appears in some of the previously mentioned coefficients, the dynamics are now expressed with linear ordinary differential equations.

3. Linear Theory Solution

Linear theory offers physical insight into the flight dynamics because closed-form solutions can be readily obtained.[4] Because V changes slowly with respect to the other variables, it is thus considered constant, $V \approx V_0$, when it appears as a coefficient in all dynamic equations except its own. Moreover, pitch attitude of the projectile is regarded as constant in the velocity equation, thus decoupling the velocity equation. The epicyclic motion, equation 28, together with the roll dynamics, equation 27, is uncoupled and forms a linear system of equations. In projectile linear theory, the Magnus force in equations 24 and 25 is typically regarded as small so that in further manipulation of the equations, all Magnus forces will be dropped. However, it is important to retain Magnus moments due to the fact that a cross product between Magnus force and its respective moment arm is not necessarily small.

The solution to the differential equation 26, for the forward velocity, is

$$V(s) = \sqrt{V_0^2 e^{-2a_V s} + \frac{b_V}{a_V}\left(e^{-2a_V s} - 1\right)}. \tag{30}$$

When $\theta_0 = 0$, the velocity solution reduces to the familiar exponential decay form.[4] The roll dynamic equation is a nonhomogeneous linear differential equation with the following solution for $\theta_0 = 0$:

$$p(s) = \left(p_0 + \frac{a_p V_0}{b_p + a_V}\right)e^{b_p s} - \frac{a_p}{b_p + a_V} V. \tag{31}$$

Noting that $p(s) = \phi' V/D$ implies

$$\phi = \frac{(\phi'_\infty - \phi'_0)\left(e^{-K_p s} - 1\right)}{K_p} + \phi'_\infty s. \tag{32}$$

Neglect the product of damping and the product AE since the density ratio is assumed small. Defining $\xi \equiv (\tilde{v} + i\tilde{w})/V_0$ enables equation 28 to be reduced to a single differential with the following form,[4]

$$\xi'' - (E - A)\xi' - (AE + C)\xi = -(MM + EFF)\,e^{i(\phi + \phi)} - iEG/V_0, \tag{33}$$

by assuming p is small and ignored.

Rather than solve for the lateral translation and rotational velocity components, via equation 33, a more direct way to obtain the effects of asymmetry is to solve the swerve differential equations. The lateral translation and rotational velocity components are contained in the attitude differential equations, and the attitudes are contained within the swerve differential equations.

4. Swerve

Swerving motion along the earth-fixed J_I and K_I axes results from a combination of the normal aerodynamic forces, as the projectile pitches and yaws, plus the forces and moments due to the configuration asymmetry. Differentiating equations 22 and 23, with respect to nondimensional arc length and using the definition of ξ with equation 33, leads to the following expression,

$$\frac{(AE+C)}{D}(y''+iz'') = -(AMM-CFF)e^{i(\phi+\phi_B)} + \frac{iCG}{V_0} + (E-A)A\xi' - A\xi''. \tag{34}$$

For a stable projectile, the swerve caused by epicyclical vibration decays as the projectile progresses downrange and does not affect the long-term lateral motion. However, the assumption that the projectile is configurationally asymmetric causes an integrated effect that contributes to the long-term lateral motion of the projectile. Linear theory shows this center of mass motion contains terms that are bounded with arc length s plus terms that are linear with s and with the inclusion of gravity the solution of equation 34 will have even higher order diverging terms. These higher order terms are typically denoted as gravity drop. The linear terms are called jump terms, which are caused by initial conditions at the gun muzzle, forces caused by asymmetry, and aerodynamic characteristics. Ignoring gravity and evaluating the following limits formally defines aerodynamic jump

$$\lim_{s\to\infty}\frac{y(s)}{Ds} = \Gamma_J, \text{ and } \lim_{s\to\infty}\frac{z(s)}{Ds} = \Gamma_K. \tag{35}$$

The total aerodynamic jump vector Γ is expressed as the sum of two vectors. The first vector represents the muzzle conditions and the second results from asymmetry subjected to a varying roll rate:

$$\begin{Bmatrix}\Gamma_J\\ \Gamma_K\end{Bmatrix} = \frac{A}{(AE+C)V_0}\begin{Bmatrix}FF\,V_0\cos(\phi_B) - v_0E - r_0D\\ FF\,V_0\sin(\phi_B) - w_0E + q_0D\end{Bmatrix} + \begin{Bmatrix}\mathrm{Re}\,\Pi\\ \mathrm{Im}\,\Pi\end{Bmatrix}, \tag{36}$$

for which

$$\Pi = -\frac{AMM - CFF}{AE+C}\lim_{s\to\infty}\frac{1}{s}\int_0^s\int_0^\tau e^{i\phi(\sigma)}\,d\sigma\,d\tau. \tag{37}$$

The quantity Π is the contribution to the jump vector attributed to the assumed asymmetry of the projectile. The appendix shows that

$$\Pi = -\frac{A\,MM - C\,FF}{A\,E + C}\left[\frac{i}{Z} + \frac{\Gamma}{1 - i\,Z}\ {}_1F_1\left(1;2 - i\,Z, -i\,\Gamma Z\right)\right], \tag{38}$$

where ${}_1F_1$ is the confluent hypergeometric[5] function. Apparently, neither Murphy and Bradley[1] nor Fansler and Schmidt[2] were aware of equation 38, which makes their limiting and asymptotic analysis, $\Gamma = 1$ and $|\Gamma - 1| << 1$, unnecessary for calculations of jump due to asymmetric configurations.

For constant rolling motion $\Gamma = 0$ and for comparison purposes, the following definition introduced by Murphy and Bradley[1] will be used here, $\hat{\Pi} = -i\,\Pi\,Z$, so that $\hat{\Phi} = -i\,\Phi\,Z$ of equation (A-6) of the appendix now becomes

$$\begin{aligned} \hat{\Phi} &= 1 - i\Gamma Z \int_0^1 e^{-iZ\left(\Gamma y - \ln(1-y)\right)}\,dy \\ &= 1 - i\Gamma Z\left(1 - iZ\right)^{-1}\ {}_1F_1\left(1;2 - Zi, -i\Gamma Z\right). \end{aligned} \tag{39}$$

Polar plots of results from equation 39 composed to Murphy and Bradley[1] are given in figure 3, for $|\hat{\Phi}|$, and figure 4 gives the argument $\hat{\Phi}$ from equation 39.

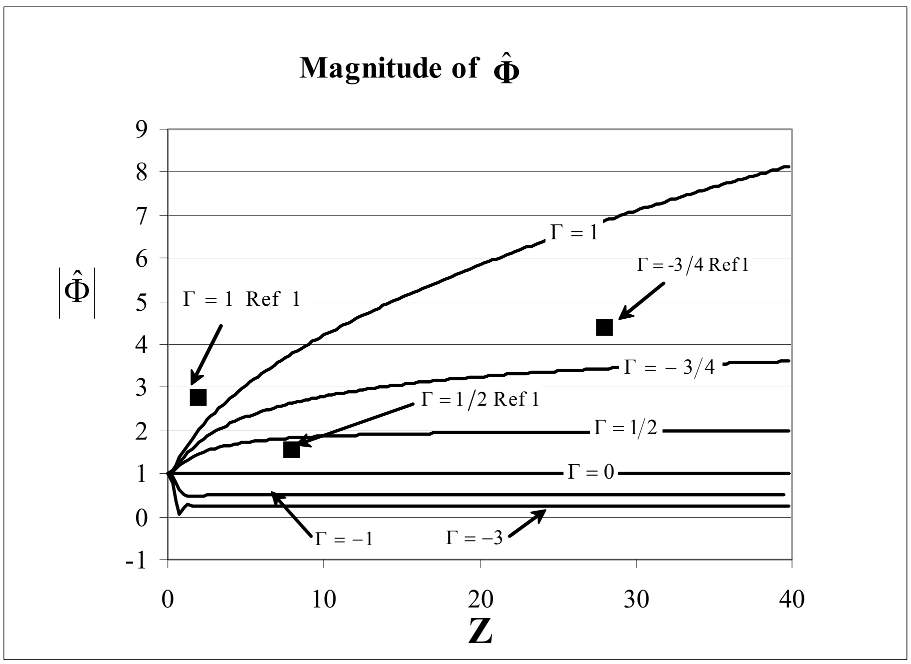

Figure 3. Magnitude of $\hat{\Phi}$ as a function of Z.

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[5] Abramowitz, M.; Stegun, I. A., Eds. *Handbook of Mathematical Functions,* Series 55; National Bureau of Standards Applied Mathematics: Dover Publishing Co., 1967.

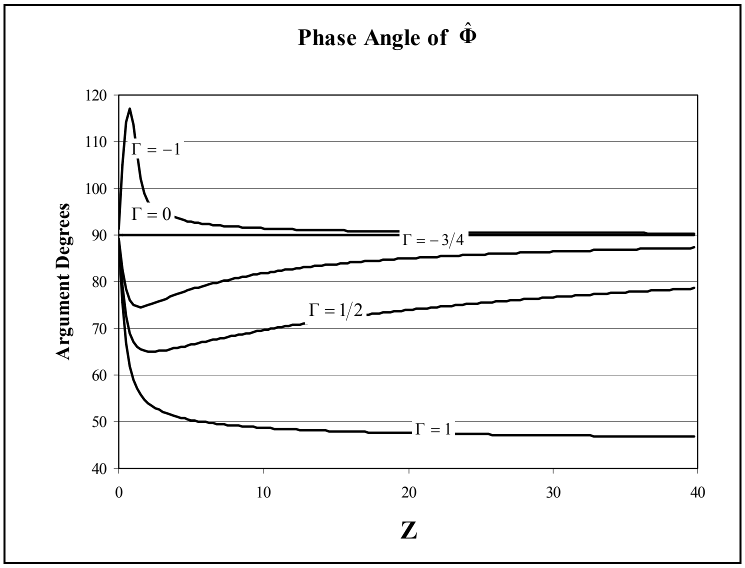

Figure 4. Argument of $\hat{\Phi}$ as a function of Z.

Successive integration by parts of the last equation shows

$$\hat{\Phi} = 1 - \frac{\Gamma}{\Gamma - 1} + \frac{i\Gamma}{Z(\Gamma-1)^3} - \frac{\Gamma(2\Gamma+1)}{Z^2(\Gamma-1)^5} + \frac{i\Gamma}{Z^2}\int_0^1 F_2' \, e^{-iZ(\Gamma y - \ln(1-y))} \, dy, \tag{40}$$

where

$$F_0 = \frac{i(y-1)}{Z(\Gamma(y-1)+1)} \text{ and } F_n = F_0 \, F_{n-1}'.$$

Hence, in the limit of large,

$$Z \to \infty, \ \hat{\Phi} \to \frac{1}{1-\Gamma} = \frac{\phi_\infty'}{\phi_0'} . \tag{41}$$

Furthermore, equation 39 can be expanded using a Kummer series giving the following expression:

$$\hat{\Phi} = 1 - \frac{iZ\Gamma}{1-iZ} - \frac{Z^2\Gamma^2}{(1-iZ)(2-iZ)} + \frac{iZ^3\Gamma^3}{(1-iZ)(2-iZ)(3-iZ)} + \frac{Z^4\Gamma^4}{(1-iZ)(2-iZ)(3-iZ)(4-iZ)} - \cdots \tag{42}$$

The limit of equation 42, $Z \to \infty$, is

$$\hat{\Phi} = 1 + \Gamma + \Gamma^2 + \Gamma^3 + \Gamma^4 \cdots = \frac{1}{1-\Gamma} \text{ for } \Gamma < 1. \tag{43}$$

This last result, in light of the general limiting case of equation 41, suggests the Kummer series, equation 42, when transformed to a continued fraction, may produce an accurate approximation to $\hat{\Phi}$ for all values of Γ. To continue the investigation, the assumption was made that a reasonable approximation has the following representation:

$$\hat{\Phi} = a0\Gamma \Big/ \Big(1 + a1\Gamma \Big/ \Big(1 + a2\Gamma \Big/ \Big(1 + a3\Gamma \Big/ \cdots \Big(1 + a7\Gamma\Big)\Big)\Big)\Big)\Big)\Big). \tag{44}$$

After finding the coefficients $a0, a1, a2 \cdots a7$ results in the rational approximation

$$\hat{\Phi} = \frac{\begin{array}{l} iZ^5(Z+2i)(Z+3i)(Z+4i)(Z+5i)^2(Z+6i)(Z+7i)(Z(Z+5i)-10)\Gamma^4 \\ -iZ^3(Z+2i)(Z+3i)(Z+4i)^2(Z+5i)^2(Z+6i)^2(Z+7i)(Z(3Z+5i)-8)\Gamma^3 \\ +3iZ^3(Z+2i)(Z+3i)^2(Z+4i)^2(Z+5i)^2(Z+6i)^2(Z+7i)^2\Gamma^2 \\ -iZ(Z+2i)^2(Z+3i)^2(Z+4i)^2(Z+5i)^2(Z+6i)^2(Z+7i)^2(Z+8i)\Gamma \end{array}}{\begin{array}{l} i(Z+i)(Z+2i)^2(Z+3i)^2(Z+4i)^2(Z+5i)(Z+6i)(Z+7i)Z \\ \left[\begin{array}{l} Z^3\Gamma^4 - 4Z(Z+5i)\Gamma^3 + 6Z(Z+5i)(Z+6i)\Gamma^2 - 4(Z+5i)(Z+6i)(Z+7i)\Gamma + \\ (Z+13i)(Z(Z+13i)-82) \end{array}\right] + 1680 \end{array}}. \tag{45}$$

Taking the limit of this expression, for $Z \to \infty$, yields $\hat{\Phi} \to \dfrac{1}{1-\Gamma}$, which indicates equation 45 is a reasonable approximation when $\Gamma \neq 1$. Figure 5 shows some comparisons between the exact solution, equation 39, and its rational approximation equation 45 for various values of Γ. It is noteworthy that even for the case where $\Gamma = 1$, the comparison shows good agreement for $Z < 20$.

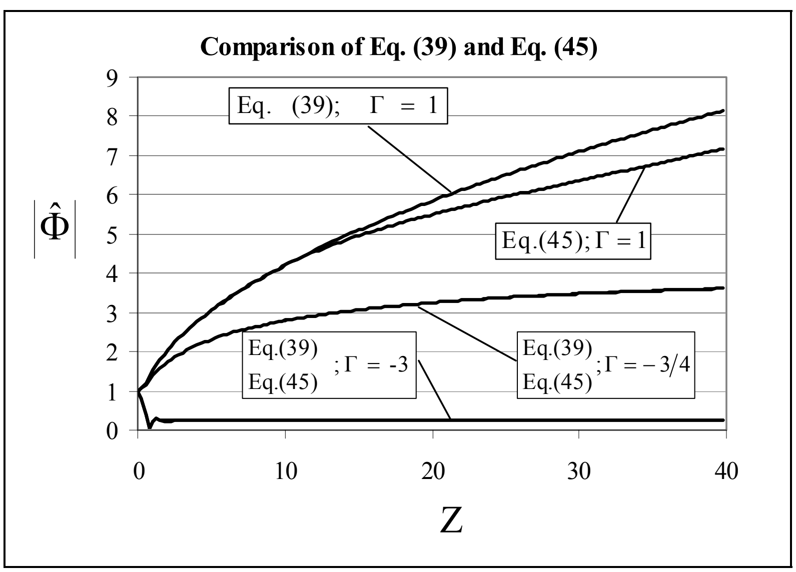

Figure 5. A comparison of the exact vs. a rational approximation of $\hat{\Phi}$.

5. Conclusions

Previous efforts describing jump due to asymmetry did not develop the closed-form solution presented here. All of the analysis is based on projectile linear theory, which leads to an expression based on the confluent hypergeometric function ${}_1F_1$. This solution is well approximated, for the arguments used here, with a simple rational expression obtained from a continued fraction expansion of the closed-form solution. This is a further extension of the Murphy and Bradley[1] and Fansler and Schmidt[2] results (see figure 3) that will prove useful for analysis and design purposes.

Appendix. Integral

Let $\Gamma = 1 - \phi_0' / \phi_\infty'$, $Z = \phi_\infty' / K_p$ and $\tau = sK_p$, then equation 32 will be written as

$$\phi(\tau) = Z\left[\tau + \Gamma\left(e^{-\tau} - 1\right)\right]. \tag{A-1}$$

Then

$$\begin{aligned}\Phi &= \lim_{s\to\infty} \frac{1}{s}\int_0^s\int_0^\tau e^{i\phi(\sigma)}\, d\sigma\ d\tau \\ &= \lim_{s\to\infty} \frac{1}{s}\int_0^s (s-\tau)\, e^{iZ\left[\tau + \Gamma\left(e^{-\tau}-1\right)\right]}\, d\tau \quad .\end{aligned} \tag{A-2}$$

Ignoring the limiting process for the moment allows the last equal sign to be written as

$$s\Phi = e^{-iZ\Gamma}\sum_{n=0}\frac{(iZ\Gamma)^n}{n!}\int_0^s (s-\tau)\, e^{(iZ-n)\tau}\, d\tau. \tag{A-3}$$

Integrating the last expression and taking the limit $s\to\infty$ causes equation A-3 to become

$$\Phi = e^{-iZ\Gamma}\left[\frac{i}{Z} + \sum_{n=1}\frac{(iZ\Gamma)^n}{(n - iZ)\, n!}\right], \tag{A-4}$$

where after writing the summation as two sums over even and odd values of n respectively becomes

$$\Phi = e^{-iZ\Gamma}\left[\begin{array}{l} \qquad\qquad \dfrac{i}{Z} \\ +\displaystyle\sum_{n=1}\frac{(-1)^n (Z\Gamma)^{2n}}{(2n)!}\int_0^1 t^{(2n-1-iZ)}\, dt \\ +i\displaystyle\sum_{n=0}\frac{(-1)^n (Z\Gamma)^{2n+1}}{(2n+1)!}\int_0^1 t^{(2n-iZ)}\, dt \end{array}\right]. \tag{A-5}$$

Using Taylor Series expansions followed with partial integration given by A and S[5],

$$\Phi = \frac{i}{Z} + \Gamma \int_0^1 e^{iZ\Gamma(t-1)}\, t^{-iZ}\, dt = \frac{i}{Z} + \Gamma \int_0^1 e^{-iZ\Gamma y} \left(1-y\right)^{-iZ} dy$$

$$= \frac{i}{Z} + \frac{\Gamma}{1 - iZ}\, {}_1F_1\left(1; 2 - iZ, - iZ\Gamma\right). \qquad \text{(A-6)}$$

List of Symbols, Abbreviations, and Acronyms

a_p	Integration constant $a_p = \dfrac{\rho C_{LDD} D^2 S}{2 I_x}$.
a_v	Integration constant $a_v = \dfrac{\rho C_{x0} D S}{2m}$.
b_p	Integration constant $b_p = \dfrac{\rho C_{LP} D^3 S}{4 I_x}$.
b_v	Integration constant $b_v = \dfrac{g \theta_0 D}{a_v}$.
C_i	Projectile aerodynamic coefficients.
D	Projectile characteristic length (diameter).
FF	Force component caused by asymmetry.
g	Gravitational constant.
G	Scaled gravitational constant $G = gD/V_0$.
I_X I_Y	Mass moments of inertia.
K_p	Constant defined as $K_p = -\dfrac{\rho D S \left(2 C_{X0} I_x + m C_{LP} D^2 \right)}{4 m I_x}$.
L $\tilde{M}$ $\tilde{N}$	Applied moments about projectile mass center expressed in the no-roll frame.
MM	Moment component caused by asymmetry.
m	Projectile mass.
p $\tilde{q}$ $\tilde{r}$	Angular velocity components vector of projectile in the no-roll frame.
s	Dummy integration variable.

S	Surface area $S = \pi D^2 / 4$.
u v w	Mass center velocity components in the body reference frame.
V_0	Forward velocity of projectile.
u $\tilde{v}$ $\tilde{w}$	Mass center velocity components in the no-roll reference frame.
$\begin{Bmatrix} x \\ y \\ z \end{Bmatrix}$	Position vector of body center of mass in an inertial reference frame.
Z	Substitution variable defined as $Z = \phi'_\infty / K_p$.
α	Longitudinal aerodynamic angle of attack.
β	Lateral aerodynamic angle of attack.
Γ_J Γ_K	$\begin{matrix} J_I \\ K_I \end{matrix}$ Components of aerodynamic jump.
Γ	Substitution variable defined as $\Gamma = 1 - \phi'_0 / \phi'_\infty$.
Π	Complex aerodynamic jump caused by asymmetry.
ϕ θ ψ	Euler roll, pitch, and yaw angles of the projectile.
ϕ'_0 ϕ'_∞	Euler roll rates $\begin{matrix} \phi'_0 \\ \phi'_\infty \end{matrix} = \begin{matrix} p_0 D / V_0 \\ a_p D / K_p \end{matrix}$.
ϕ_B	Euler roll angle of configuration asymmetry.
ρ	Air density.

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